u_e, equilibrium moisture content of material, kg/kg; l, length of drying chamber, m; γ, concentration of dry substance in moist disperse material, kg/kg.

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EFFICIENCY OF COOLING THERMOELECTRIC ELEMENTS OF ARBITRARY SHAPE

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The problem of the limiting efficiency of thermoelectric cooling is considered in the general case when no limitations are imposed on the shape of the thermoelectric elements and their contact surfaces.

It is well known that the permissible temperature drop and the limiting power efficiency of thermoelectric elements of prismatic shape are uniquely determined by the figure of merit of the thermoelectric materials and the temperature level at which the elements operate and are independent of their geometrical dimensions [1]. It is of considerable interest to clarify what this behavior is in the general case when no limitations are imposed on the shape of the thermoelectric element and on its contact surfaces.

Consider a thermoelectric element (see Fig. 1) having two contact surfaces s_0 and s_1. We will assume that the heat exchange between the thermoelectric element and the external sources only occurs over the surfaces of the contacts, which are simultaneously isothermal and equipotential, while the remaining surface of the thermoelectric element is adiabatically and electrically insulated. We will consider the properties of the temperature field which is established when a potential difference $u_1 - u_0$ is applied, and we will determine the heat flow entering the contact surfaces along the body of the thermoelectric element.

If we ignore the temperature dependence of the physical parameters of the thermoelectric material, the temperature field inside the region v bounded by the surface s of the thermoelectric element corresponds to the Poisson equation

$$\nabla^2 \vartheta = -\frac{i^2}{\lambda\sigma}. \tag{1}$$

Equation (1) is uniform and there are also the nonuniform boundary conditions:

$$\vartheta|_{s_0} = 0; \quad \vartheta|_{s_1} = T_1 - T_0; \quad \left.\frac{\partial \vartheta}{\partial n}\right|_{\Sigma} = 0. \tag{2}$$

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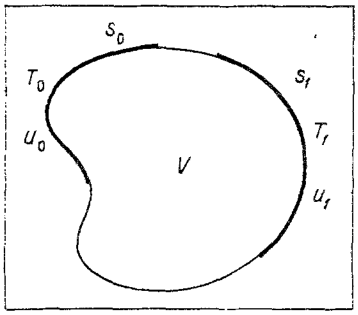

Fig. 1. Illustrating the problem of determining the flow of heat by conduction to the contact surfaces.

The solution of this equation can be represented in the form of the sum of the solution ϑ' of the corresponding uniform equation for assigned conditions (2), and the solution ϑ'' of Eq. (1) for zero boundary conditions (which corresponds to equality of the temperatures of the contacts $T_1 = T_0$):

$$\vartheta(x, y, z) = \vartheta'(x, y, z) + \vartheta''(x, y, z). \tag{3}$$

The functions $\vartheta'(x, y, z)$ and $\vartheta''(x, y, z)$ have a simple physical meaning. The first of them gives the distribution of the excess temperatures $\vartheta = T - T_0$ in the thermoelectric element when there is no current, when an assigned temperature difference $T_1 - T_0$ is maintained at the contacts, while the second gives the distribution of the excess temperatures in the same thermoelectric element when there is a current present, but for uniform contact temperatures.

In accordance with this, the heat transfer vector of thermal conduction at any point of a thermoelectric element of arbitrary shape is equal to the sum of the thermal conduction vectors for the two above-mentioned special cases of temperature distribution:

$$\mathbf{q}_F = \mathbf{q}_{F_1} + \mathbf{q}_{F_2}. \tag{4}$$

The flux of the vector $\mathbf{q}_F$ through the contact surfaces can be written in the form

$$\iint_{s_{0,1}} \mathbf{q}_F \cdot d\mathbf{s} = Q^F|_{s_{0,1}} = \iint_{s_{0,1}} \mathbf{q}_{F_1} \cdot d\mathbf{s} + \iint_{s_{0,1}} \mathbf{q}_{F_2} \cdot d\mathbf{s}. \tag{5}$$

Since the surfaces of the contacts are isothermal, the vectors $\mathbf{q}_F$, $\mathbf{q}_{F_1}$, and $\mathbf{q}_{F_2}$ are directed along the normals to the surfaces, and, consequently, the scalar products of the vectors in Eq. (5) can be replaced by products of their absolute magnitudes. Assuming that the heat-transfer vector is directed toward the side in which the temperature decreases, while the vector of the small area ds is directed toward the external normal, we can write

$$\int q_F ds = Q_F|_{s_0} = \iint_{s_0} q_{F_1} ds + \iint_{s_0} q_{F_2} ds, \quad -\iint_{s_1} q_F ds = -Q_F|_{s_1} = -\iint_{s_1} q_{F_1} ds + \iint_{s_1} q_{F_2} ds. \tag{6}$$

Consider the integrals on the right side of Eq. (6). Since the flux $\mathbf{q}_{F_1}$ corresponds to the case when there are no internal heat sources, we have the obvious equation

$$\iint_{s_0} q_{F_1} ds = \iint_{s_1} q_{F_1} ds = Q_{F_1}. \tag{7}$$

For the case when there is a flow of heat by thermal conduction $\mathbf{q}_{F_2}$ in the body due to internal sources of Joule heat dissipation of overall power Q_J, we can write, in accordance with Gauss's theorem,

$$\iint_{s} \mathbf{q}_{F_2} \cdot d\mathbf{s} = \iint_{s_0} q_{F_2} ds + \iint_{s_1} q_{F_2} ds = Q_J. \tag{8}$$

Assuming that a certain fraction φ of the Joule heat flows to the contact s_0, we have

$$\iint_{s_0} q_{F_2} ds = \varphi Q_J; \quad \iint_{s_1} q_{F_2} ds = (1 - \varphi) Q_J. \tag{9}$$

Taking Eqs. (7) and (9) into account, Eqs. (6) take the form

$$Q_F|_{s_0} = Q_{F_1} + \varphi Q_J, \quad Q_F|_{s_1} = Q_{F_1} - (1-\varphi) Q_J. \tag{10}$$

We will determine the fraction φ of the total power of internal heat dissipation transferred by thermal conduction to the surface s_0 when both contacts of the thermoelectric element are maintained at the same temperatures $T_0 = T_1$. Note that the potential distribution in a nonisothermal conductor is given by the generalized Ohm's law [3]

$$\nabla u = -\frac{\nabla\mu^*}{e} - \alpha\nabla T - \frac{1}{\sigma}\mathbf{i}. \tag{11}$$

It is seen from Eq. (11) that in a uniform isotropic medium, the physical parameters of which are independent of the temperature, the field of stationary flows is a potential field, i.e., we can write

$$\mathbf{i} = -\frac{\sigma}{e}\nabla\mu. \tag{12}$$

The potential $\mu = \mu^* + eu + e\alpha T$ characterizing this field, because of its structure and physical meaning, can be called thermoelectrochemical.

The potential field μ and the temperature field T in the thermoelectric element are closely connected to one another, and for the Joule heat distribution this relation is of considerable importance. Hence, to solve the above problem it is important to establish the properties of the fields and their interaction. We will use for this purpose Green's formula for the functions μ and T'':

$$\iiint_v (\mu\nabla^2 T'' - T''\nabla^2\mu)\, dv = \iint_s \mu\frac{\partial T''}{\partial n}\, ds - \iint_s T''\frac{\partial\mu}{\partial n}\, ds. \tag{13}$$

It follows directly from Eq. (12) that when $\sigma = \text{const}$, $\nabla\mathbf{i} = \nabla^2\mu = 0$. In addition, since the external contact surfaces of the thermoelectric elements are adiabatically and electrically insulated, everywhere outside of the contacts the integrals on the right side of Eq. (13) vanish. Taking these facts into account, and also the condition $T_1 = T_0$, it is easy to convert Eq. (13) to the form

$$\iiint_v \mu\nabla^2 T''\, dv = \mu_1 \iint_{s_1} \frac{\partial T''}{\partial n}\, ds + \mu_0 \iint_{s_0} \frac{\partial T''}{\partial n}\, ds. \tag{14}$$

Putting the operator $\nabla^2 T''$ equal to its value $-i^2/\lambda\sigma$ on the left side of Eq. (14) and multiplying both sides by $-\lambda$, we obtain

$$\frac{1}{\sigma}\iiint_v \mu i^2 dv = \mu_1 \iint_{s_1} q_{F_2} ds + \mu_0 \iint_{s_0} q_{F_2} ds \tag{15}$$

or, taking Eqs. (9) and (12) into account,

$$\frac{\sigma}{e^2}\iiint_v \mu(\nabla\mu)^2 dv = [\mu_1(1-\varphi) + \mu_0\varphi]\, Q_J. \tag{16}$$

The elementary volume dv can be considered as part of a tube of current of infinitely small cross section ds_μ contained between two infinitely close surfaces of equal potential. Denoting the left side of Eq. (16) by j, we obtain

$$j = \frac{\sigma}{e^2}\iiint_v \mu\left(\frac{\partial\mu}{\partial n}\right)^2 dn\, ds_\mu = \frac{\sigma}{e^2}\iiint_v \mu\frac{\partial\mu}{\partial n}\, ds_\mu d\mu. \tag{17}$$

The latter integral can be obtained by summing initially the values of the function under the integral over the whole surface μ = const and then integrating from μ_0 to μ_1:

$$j = \int_{\mu_0}^{\mu_1} \frac{\mu}{e} \iint_{s_\mu} \frac{\sigma}{e}\frac{\partial\mu}{\partial n}\, ds_\mu d\mu = \int_{\mu_0}^{\mu_1} \frac{\mu}{e} \iint_{s_\mu} \mathbf{i}\cdot ds_\mu d\mu = I\int_{\mu_0}^{\mu_1} \frac{\mu}{e}\, d\mu = I\,\frac{\mu_1 - \mu_0}{e}\,\frac{\mu_1+\mu_0}{2}. \tag{18}$$

Note that in view of the equation $T_1 = T_0$ in the case considered $\mu_1^* = \mu_0^*$, and, consequently,

$$j = I(u_1 - u_0)\frac{\mu_1 + \mu_0}{2} = Q_J\frac{\mu_1 + \mu_0}{2} . \qquad (19)$$

Equating the right sides of Eqs. (16) and (19), we obtain after some simple algebra $\varphi \equiv 1/2$. Hence, when $T_0 = T_1$, irrespective of the shape of the thermoelectric element and its contact surfaces, half of the total power of the internal sources of Joule heat are applied to each of the contacts. If the temperatures of the contacts are different, then, as follows from Eq. (10), the total heat flux to the points of contact is equal to the algebraic sum of the heat conduction flux when there is no current and half the power of the internal sources. This important conclusion enables us to analyze the efficiency of a thermoelectric element as a source of cooling in the most general case.

Consider the thermal balance of the thermoelectric contacts. The thermal fluxes Q_0 and Q_1, which the thermoelectric element exchanges with the external sources, are given by the relations [2]

$$Q_0 = Q_\pi|_{s_0} - Q_F|_{s_0}, \quad Q_1 = Q_\pi|_{s_1} - Q_F|_{s_1} . \qquad (20)$$

Substituting Eq. (10) into Eq. (20) and assuming $\varphi = 1/2$, we obtain

$$Q_0 = Q_\pi|_{s_0} - \frac{1}{2}Q_J - Q_{F_1}, \quad Q_1 = Q_\pi|_{s_1} + \frac{1}{2}Q_J - Q_{F_1}. \qquad (21)$$

Hence, the form of the equations of heat balance of the contacts in the general case has the same form as for thermoelectric elements of the simplest geometrical shape [1, 2].

Relations (21) can also be represented in the form

$$\begin{aligned} Q_0 &= \alpha T_0 I - \frac{1}{2}\frac{I^2}{\sigma\Phi} - \lambda\Phi(T_1 - T_0), \\ Q_1 &= \alpha T_1 I + \frac{1}{2}\frac{I^2}{\sigma\Phi} - \lambda\Phi(T_1 - T_0). \end{aligned} \qquad (22)$$

The products $\lambda\Phi$ and $\sigma\Phi$ occurring here are given by the obvious relations

$$\lambda\Phi = \frac{Q_{F_1}}{T_1 - T_0}, \quad \sigma\Phi = \frac{I}{u_1 - u_0} \qquad (23)$$

and have the physical meaning of the thermal and electrical conductivities of the thermoelectric element.

The quantity Φ is entirely determined by the shape of the thermoelectric element and, in view of the proportionality of the thermal and electrical conductivities, has the same numerical values in relations (23). In particular, for a prismatic thermoelectric element in which the temperature distribution (when there is no electric current) and the potential distribution obey a linear law, the form factor Φ is equal to the ratio of the cross-sectional area of the thermoelectric element to its length.

Relations (22) have exactly the same form as the well-known equations for the thermal fluxes at the contacts of the usual prismatic thermoelectric element. Consequently, we can state that all the limitations on the permissible temperature drop and energy efficiency established previously for the prismatic thermoelectric element [1, 2], also hold for a thermoelectric element of arbitrary shape if its contact surfaces are isothermal and equipotential, while the remaining surface is thermally and electrically insulated. In other words, irrespective of the shape of the thermoelectric element, its limiting efficiency is uniquely determined by the dimensionless quantity zT, the figure of merit of the material. This conclusion holds for a uniform isotropic thermoelectric material, assuming that its physical parameters are independent of the temperature. An attempt to estimate the effect of the shape of the thermoelectric element on its energy efficiency in the more general case when the properties of the semiconducting materials are given in the form of arbitrary functions of the coordinates and temperature has been made in [4]. However, in that publication it was assumed that the electric field in a nonuniform thermoelectric element is a potential field (the vector of the current density is proportional to the gradient of a certain scalar function having the meaning of potential). This assumption is not justified in reality. Physical and thermal nonuniformities give rise in the thermoelectric element to extraneous emfs, and the electric field ceases to be a potential field [5]. Hence, the conclusions stated in [4] that the maximum energy efficiency in the most general case is independent of the shape of the conductor must be regarded as unproven.

NOTATION

s, total surface of the thermoelectric element; s_0, s_1, surfaces of the cold and hot contacts, respectively; Σ, thermoelectric element surface outside the contacts; v, volume of the thermoelectric element; u, electric potential; u_0, u_1, potentials of the cold and hot contacts; $\mathbf{i}$ current density vector; σ, λ, electrical conductivity and thermal conductivity; α, absolute thermal emf; $z = \alpha^2\sigma/\lambda$; T, absolute temperature; T_0, T_1, temperatures of the cold and hot contacts; T', temperature at an arbitrary point on the thermoelectric element with no current and at $T_0 \neq T_1$; T'', temperature at an arbitrary point on the thermoelectric element with current $T_0 = T_1$; $\vartheta = T - T_0$; $\vartheta' = T' - T_0$; $\vartheta'' = T'' - T_0$; ∇^2, Laplace operator; ∇, Hamiltonian operator; $q_F = -\lambda\nabla T$, conduction heat-transfer vector; $q_{F_1} = -\lambda\nabla T'$; $q_{F_2} = -\lambda\nabla T''$; Q_{F_1}, heat conduction through contact surfaces with no current; Q_J, total power of internal Joule heat sources; φ, fraction of the total power of the internal sources transferred by heat conduction to the surface s_0 at $T_0 = T_1$; $\mu = \mu^* + eu + e\alpha T$; μ^*, chemical potential; e, carrier charge $I = \iint_{s\mu} \mathbf{i} \cdot d\mathbf{s}$, electric current; s_μ, equipotential surface; $Q_{\pi}|_{s_0} = \alpha T_0 I$, Peltier heat absorbed on a cold contact; $Q_{\pi}|_{s_0} = \alpha T_1 I$, Peltier heat generated at a hot contact; Φ, form factor; Q_0, heat removed from cold source; Q_1, heat supplied to hot source.

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GENERALIZED STATIC VOLT–AMPERE CHARACTERISTICS OF THERMORESISTORS

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Similarity criteria are obtained for static volt–ampere characteristics of thermoresistors and for thermoresistors included in a circuit. A technique is described for a simplified graphic-analytical design of a circuit with a thermoresistor and rules are given for modeling thermoresistors where the dissipation coefficient varies.

1. Similarity Criteria for Static Volt–Ampere Thermoresistor Characteristics

We begin with the assumption that the temperature T is constant over the entire volume of the thermoresistor, which is approximately true [1, 2] when

$$Bi \ll 1 \qquad (1)$$

(Bi is the Biot number).

We can write the heat-balance equation for a thermoresistor, relating the current i and the voltage u on it with the environment temperature T_e and the dissipation coefficient H:

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